A REMARK ON SPACES OF AFFINE CONTINUOUS FUNCTIONS ON A SIMPLEX

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Abstract. We present an example of an infinite dimensional separable space of affine continuous functions on a Choquet simplex that does not contain a subspace linearly isometric to c. This example disproves a result stated in [8].

1. Introduction and Preliminaries

In the *Concluding remarks* in [8], the author claims that a separable predual of an abstract L_1 space contains a (complemented) copy of c (the Banach space of real convergent sequences) if its unit ball has an extreme point. Only a sketch of the proof of this property was indicated. In particular there is no proof that the extreme point and the sequence $\{y_n\}$ (1-equivalent to the standard basis of c_0 built in the main theorem of [8]), span a subspace isometric to c. The aim of this paper is to present a simple example that disproves Zippin's claim. Moreover, in the last section of our paper we point out that our example also shows that some known results, establishing geometrical properties of polyhedral Banach spaces, are incorrect.

Let B_X (S_X) denote the closed unit ball (sphere) in a real Banach space X and X^* denotes the dual of X. If K is a compact, convex subset of a linear topological space, then by $\operatorname{Ext} K$ we denote the set of all extreme points of K. A convex subset F of B_X is called a *face* of B_X if for every $x, y \in B_X$ and $\lambda \in (0,1)$ such that $(1-\lambda)x + \lambda y \in F$ we have $x, y \in F$. A face F of B_X is named a *proper face* if $F \neq B_X$. Here c denotes the Banach space of all real convergent sequences and $A(K)$ stands for a *simplex space*, that is, the space of all affine continuous functions on a Choquet simplex K endowed with the supremum norm. It is well known that $c^* = \ell_1$ and the duality is given by:

$$f(x) = f(1) \lim x(i) + \sum_{i=1}^{+\infty} f(i+1)x(i)$$

where $f = (f(1), f(2), \dots) \in \ell_1$ and $x = (x(1), x(2), \dots) \in c$. A Banach space X is called an L_1-*predual space* or a *Lindenstrauss space* if its dual is isometric to $L_1(\mu)$ for some measure μ. It is well known that this class includes all the simplex spaces. Moreover a Lindenstrauss space X is isometric to a simplex space if and only if B_X has at least one extreme point (see [7]). Finally, we recall that a Banach space X is polyhedral if the unit balls of all its finite-dimensional subspaces are polytopes (see [4]).

2. A simplex space not containing c

We begin by providing a necessary condition for the presence of a copy of c in a separable Banach space X.

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Theorem 2.1. *Let X be a separable Banach space. If X contains a subspace linearly isometric to c, then there exist $x \in X$ and a sequence $(e_n^*) \subset$ Ext B_{X^*} such that (e_n^*) is w^*-convergent to e^*, $e_n^*(x) = e^*(x) = \|e^*\| = \|x\| = 1$ and $\|e_n^* \pm e^*\| = 2$ for every $n \in \mathbb{N}$.*

Proof. Assume that X contains an isometric copy of c. Consider

$$x = (1, 1, \dots, 1, \dots) \in c,$$

the sequence $(x_n)_{n\in\mathbb{N}} \subset B_c$ defined by

$$\begin{aligned}
x_1 &= (-1, 1, 1, \dots, 1, \dots),\\
x_2 &= (1, -1, 1, 1, \dots, 1, \dots),\\
x_3 &= (1, 1, -1, 1, 1, \dots, 1, \dots),\\
&\dots
\end{aligned}$$

and the sequence $(x_n^*)_{n\in\mathbb{N}} \subset B_{c^*}$ defined by

$$\begin{aligned}
x_1^* &= (0, 1, 0, 0, \dots, 0, \dots),\\
x_2^* &= (0, 0, 1, 0, 0, \dots, 0, \dots),\\
x_3^* &= (0, 0, 0, 1, 0, 0, \dots, 0, \dots),\\
&\dots \quad .
\end{aligned}$$

Let $\widetilde{x_n^*}$ denote a norm preserving linear extension of x_n^* to the whole X. Next, let us define the sets F_n, $n \in \mathbb{N}$, by

$$F_n = \{x^* \in B_{X^*} : x^*(x_n) = -1 \text{ and } x^*(x_m) = x^*(x) = 1 \text{ for } m \neq n\}.$$

It is easy to see that

(a) $F_n \neq \emptyset$ for every $n \in \mathbb{N}$ (because $\widetilde{x_n^*} \in F_n$),
(b) F_n is a w^*-closed proper face of B_{X^*}, for every $n \in \mathbb{N}$,
(c) $F_n \cap F_m = \emptyset$ provided $m \neq n$.

Hence, Ext $F_n \neq \emptyset$ by the Krein-Milman Theorem, Ext $F_n \subset$ Ext B_{X^*} for every $n \in \mathbb{N}$ and Ext $F_n \cap$ Ext $F_m = \emptyset$ whenever $m \neq n$.

Let $e_n^* \in$ Ext $F_n \subset$ Ext B_{X^*}. We can assume that (e_n^*) is w^*-convergent, let us say to e^*. Then

(d) $e_n^*(x) = e^*(x) = \|e^*\| = \|x\| = 1$ for every $n \in \mathbb{N}$,
(e) $e^*(x_i) = \lim_n e_n^*(x_i) = 1$ for every $i \in \mathbb{N}$.

Consequently, for every $n \in \mathbb{N}$, we have

$$2 \geq \|e^* - e_n^*\| \geq e^*(x_n) - e_n^*(x_n) = 1 - (-1) = 2$$

and

$$2 \geq \|e^* + e_n^*\| \geq e^*(x) + e_n^*(x) = 1 + 1 = 2.$$

□

The previous theorem gives a corollary which allows us to show that the announced example does not contain c.

Corollary 2.2. *Let X be a predual of ℓ_1. If X contains a subspace isometric to c then there exist $x \in B_X$ and a subsequence $(e_{n_k}^*)_{k\in\mathbb{N}}$ of the standard basis $(e_n^*)_{n\in\mathbb{N}}$ in ℓ_1 such that*

(1) *$e_{n_k}^* \overset{\sigma(\ell_1, X)}{\longrightarrow} e^*$ and supp $e_{n_k}^* \cap$ supp $e^* = \emptyset$ for every $k \in \mathbb{N}$, where for $x^* \in \ell_1 = X^*$ we put supp $x^* := \{i \in \mathbb{N} : x^*(i) \neq 0\}$,*
(2) *$e_{n_k}^*(x) = e^*(x) = 1$ for every $k \in \mathbb{N}$.*

Example 2.1. *Let*

$$W = \left\{ x = (x(1), x(2), \dots) \in c : \lim_i x(i) = \sum_{i=1}^{\infty} \frac{x(i)}{2^i} \right\}.$$

The hyperplane W has the following properties:

(a) *The map $\phi : \ell_1 \to W^*$ defined by*

$$(\phi(y))(x) = \sum_{j=1}^{+\infty} x(j)y(j),$$

where $y = (y(1), y(2), \dots) \in \ell_1$ and $x = (x(1), x(2), \dots) \in W$ is an onto isometry. Moreover, if (e_n^*) denotes the standard basis of ℓ_1, then

$$e_n^* \overset{\sigma(\ell_1, W)}{\longrightarrow} e^* = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots \right)$$

(see Theorem 4.3 in [1]).

(b) *From Corollary 2.2 we conclude that W does not contain a subspace linearly isometric to c.*

(c) *By Corollary 2 in [6] the set*

$$K = \left\{ (y(1), y(2), \dots) \in \ell_1 : \sum_{i=1}^{\infty} y(i) = 1, y(i) \geq 0, i = 1, 2, \dots \right\}$$

is an infinite dimensional $\sigma(\ell_1, W)$-closed proper face of B_{ℓ_1}.

(d) *It is easy to see that $x = (1, 1, \dots, 1, \dots) \in Ext\, B_W$. Consequently, as was observed in [7], W is isometric to $A(K)$. Nevertheless, in our special case this property can be shown directly.*

(e) *In order to prove that the space W is polyhedral we need a characterization of polyhedrality given by Durier and Papini (Theorem 2 in [2]): a Banach space X is polyhedral if and only if the set*

$$C(x) = \{y \in X : \exists \lambda > 0, \|x + \lambda(y - x)\| \leq 1\}$$

is a closed set for every $x \in S_X$. Moreover, we remark that $x \in S_W$ if and only if there exists at least one index $i_0 \in \mathbb{N}$ such that $|x(i_0)| = 1$. Then, an easy computation shows that

$$C(x) = \{y \in W : y(i) \leq 1 \text{ for } i \in I(x) \text{ and } y(j) \geq -1 \text{ for } j \in J(x)\},$$

where $I(x) = \{i : x(i) = 1\}$ and $J(x) = \{j : x(j) = -1\}$. Therefore the set $C(x)$ is closed for every $x \in S_W$.

Remark 2.3. Example 2.1 shows that property (2) in Corollary 2.2 does not imply that $c \subset X$. Also property (1) in the same corollary does not imply that $c \subset X$. Indeed, to this end is sufficient to consider a different hyperplane of c:

$$V = \left\{ x = (x(1), x(2), \dots) \in c : \lim_i x(i) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x(2i-1)}{2^i} \right\}.$$

By using Theorem 4.3 in [1] we have that $V^* = \ell_1$ and

$$e_{2n}^* \overset{\sigma(\ell_1, V)}{\longrightarrow} e^* = \left(\frac{1}{2}, 0, -\frac{1}{4}, 0, \frac{1}{8}, 0, -\frac{1}{16}, \dots \right).$$

It is easy to see that does not exist $x \in V$ satisfying the property (2) in Corollary 2.2. Therefore V does not contain an isometric copy of c. It would be desirable to understand if the simultaneous validity of conditions (1) and (2) ensures the presence of a isometric copy of c in a predual of ℓ_1, but we have not been able to do this. Nevertheless we show that the necessary condition expressed in Theorem

2.1 is not a sufficient condition in a general framework. Indeed, let us consider the space $X = \ell_1$ and the sequence $(x_n^*)_{n\in\mathbb{N}}$ in $\ell_1^* = \ell_\infty$ defined by

$$\begin{aligned} x_1^* &= (1,-1,1,1,\dots,1,\dots),\\ x_2^* &= (1,1,-1,1,1,\dots,1,\dots),\\ x_3^* &= (1,1,1,-1,1,1,\dots,1,\dots), \quad \dots. \end{aligned}$$

Then

(a) $x_n^* \overset{\sigma(\ell_\infty,\ell_1)}{\longrightarrow} x^* = (1,1,\dots,1,\dots)$,
(b) $\|x_n^* \pm x^*\| = 2$ for every $n \in \mathbb{N}$,
(c) $x_n^*, x^* \in \text{Ext } B_{\ell_\infty}$ for every $n \in \mathbb{N}$,
(d) for $e_1 = (1,0,0,\dots,0,\dots) \in \ell_1$ we have $x^*(e_1) = x_n^*(e_1) = 1$

but ℓ_1 does not contain c.

3. Final remarks

Different authors refer to Zippin's statement. We focus on a paper by Lazar that gives a characterization of polyhedral Lindenstrauss spaces.

In [5], the implication (1)⇒(3) in Theorem 3 is incorrect. Indeed, W is a polyhedral space but B_{W^*} contains an infinite dimensional w^*-closed proper face (see items (e) and (c) in Example 2.1). As a consequence of this remark we have that some of the implications stated in the theorem mentioned above reveal to be unproven. For instance, the implication (1)⇒(4) has no proof.

The result of Lazar has been subsequently used by Gleit and McGuigan in [3] to provide another characterization of polyhedral Lindenstrauss spaces. We remark that, in [3], the implication (3)⇒(1) in Theorem 1.2 is incorrect. Indeed, W does not contain an isometric copy of c and for $x = (1,1,\dots,1,\dots)$ in W and the w^*-limit $e^* = \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\right)$ of the standard basis in $\ell_1 = W^*$ we have (see items (b) and (a) in Example 2.1)

$$e^*(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 = \|x\| = \|e^*\|.$$

Also, the implication (3)⇒(1) in Corollary 2.7 is incorrect because W is a simplex space (see item (d) in Example 2.1).

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